Announcements

1) HW #3 up

2) Proof of triangle inequality for $\|\cdot\|_2$ completed in yesterday's notes

Recall: V a vector space over $\mathbb{R}$ or $\mathbb{C}$. A norm on V is a function

$$\|\cdot\| : V \to [0,\infty)$$

such that $\forall\, x, y, z \in V$,

1) $\|x\| = 0$ if and only if $x = 0_V$.

2) $\|\alpha x\| = |\alpha|\,\|x\|$ $\forall$ scalars α

3) $\|x-y\| \le \|x-z\| + \|z-y\|$ (triangle inequality)

Reduction for Triangle Inequality

If we set $u = x - z$ and $w = z - y$, then $u + w = x - y$. Using u and w, we can rewrite the triangle inequality as:

$$\boxed{\|u+w\| \leq \|u\| + \|w\|}$$

$\forall\ u, w \in V$

In examples, it is usually easier to prove the triangle inequality in its reduced form.

Example 1: (p-norm)

Let $p \in \mathbb{R}$, $1 \leq p < \infty$.

Consider $\mathbb{C}^n$ (or $\mathbb{R}^n$)

as a vector space over

$\mathbb{C}$ (or $\mathbb{R}^n$ over $\mathbb{R}$)

and define, for

$z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n (\mathbb{R}^n)$

$$\boxed{\|z\|_p = \left\{ \sum_{k=1}^{n} |z_k|^p \right\}^{1/p}}$$

Observe that $p=2$ gives us our example from last class, and is the most important value of p. The next most important value is $p=1$.

Let's show $\|\cdot\|_1$ is a norm.

1) $\|z\|_1 = 0$ if and only if $z = (\underbrace{0, 0, \dots, 0}_{n \text{ times}})$.

If $\|z\|_1 = 0$, then

$$\sum_{k=1}^{n} |z_k| = 0.$$

This implies $z_k = 0 \ \forall \ 1 \le k \le n$,

so z is the zero vector.

Conversely, $\|(0, 0, \dots, 0)\|_1$

$$= \sum_{k=1}^{n} 0 = 0.$$

2) $\|\alpha z\|_1 = |\alpha| \, \|z\|_1$

$\forall$ scalars α.

$$\|\alpha z\|_1 = \sum_{k=1}^{n} |\alpha z_k|$$

$$= |\alpha| \sum_{k=1}^{n} |z_k|$$

$$= |\alpha| \, \|z\|_1$$

3) If $w = (w_1, \dots, w_n) \in \mathbb{C}^n (\mathbb{R}^n)$, then $\|z+w\|_1 \le \|z\|_1 + \|w\|_1$

$$\|z+w\|_1 = \sum_{k=1}^{n} |z_k + w_k|$$

$$\le \sum_{k=1}^{n} \left(|z_k| + |w_k|\right)$$

by the triangle inequality for complex (or real) numbers

$$= \|z\|_1 + \|w\|_1$$

This shows $\|\cdot\|_1$ is a norm. Showing $\|\cdot\|_p$ is a norm for $p \neq 1, 2$ is more annoying.

Example 2: (∞ norm on $\mathbb{C}^n$)

For $\mathbb{C}^n$ (or $\mathbb{R}^n$) as a vector space over $\mathbb{C}$ (or $\mathbb{R}^n$ over $\mathbb{R}$), define, for $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ (or $\mathbb{R}^n$)

$$\boxed{\|z\|_\infty = \max_{1 \le k \le n} |z_k|}$$

Let's show $\|\cdot\|_\infty$ is a norm.

1) $\|z\|_\infty = 0$ if and only if $z = (\underbrace{0, 0, \ldots, 0}_{n \text{ times}})$.

If $z \neq (0, 0, \ldots, 0)$, then $\exists\, j$, $1 \leq j \leq n$, with $z_j \neq 0$.

Then $$\|z\|_\infty = \max_{1 \leq k \leq n} |z_k|$$
$$\geq |z_j| > 0.$$

Conversely, $$\|(0, 0, \ldots, 0)\|_\infty = \max_{1 \leq k \leq n} 0 = 0.$$

2) $\|\alpha z\|_\infty = |\alpha|\,\|z\|_\infty$

$\forall$ scalars α.

$$\|\alpha z\|_\infty = \max_{1 \le k \le n} |\alpha z_k|$$

$$= \max_{1 \le k \le n} \left(|\alpha| \cdot |z_k|\right)$$

$$= |\alpha| \max_{1 \le k \le n} |z_k|$$

$$= |\alpha| \cdot \|z\|_\infty$$

3) If $w=(w_1,\dots,w_n)\in\mathbb{C}^n$ (or $\mathbb{R}^n$),

$$\|z+w\|_\infty \le \|z\|_\infty+\|w\|_\infty$$

$$\|z+w\|_\infty = \max_{1\le k\le n} |z_k+w_k|$$

$$\le \max_{1\le k\le n}\left(|z_k|+|w_k|\right)$$

by the triangle inequality for $\mathbb{C}$ (or $\mathbb{R}$)

$$\le \left(\max_{1\le k\le n}|z_k|\right)+\left(\max_{1\le k\le n}|w_k|\right)$$

$$= \|z\|_\infty+\|w\|_\infty$$

This shows $\|\cdot\|_\infty$ is a norm. It is so named because,

$\forall\ z \in \mathbb{C}^n$ (or $\mathbb{R}^n$),

$$\lim_{p\to\infty} \|z\|_p = \|z\|_\infty .$$

Example 3: (∞ norm on $\ell^\infty(\mathbb{N})$)

Recall $\ell^\infty(\mathbb{N})$ is the vector space over $\mathbb{C}$ of all complex sequences $(z_k)_{k=1}^\infty$ such that for every sequence, $\exists\, M>0$ with $|z_k| < M$ $\forall\ 1 \leq k < \infty$.

We may define a norm $\|\cdot\|_\infty$ on $\ell^\infty(\mathbb{N})$ by $\|(z_k)_{k=1}^\infty\|_\infty$

$$= \sup_{k \geq 1} |z_k|$$

where "sup" = supremum.

This is a norm on $\ell^\infty(\mathbb{N})$ - but we won't prove it!

Definition: (inner product)

Let V be a vector space over $\mathbb{C}$ (or $\mathbb{R}$).

An inner product is a function

$$\langle \cdot , \cdot \rangle : V \times V \to \mathbb{C} \text{ (or } \mathbb{R})$$

such that $\forall\ x, y, z \in V$ and α scalar,

1) $\langle x,x\rangle \geq 0$ and

$\langle x,x\rangle = 0$ if and only if $x = 0_V$.

2) $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$

$\langle x, y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$

(additivity in each coordinate)

3) $\langle \alpha x, y\rangle = \alpha \langle x,y\rangle$

$\langle x, \alpha y\rangle = \overline{\alpha}\langle x,y\rangle$

2)+3) is called sesquilinearity

(sesqui = $1\frac{1}{2}$)

4) $\langle x, y\rangle = \overline{\langle y, x\rangle}$